Exp-Function Method for *N*-Soliton Solutions of Nonlinear Differential-Difference Equations

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In this paper, the exp-function method is generalized to construct *N*-soliton solutions of nonlinear differential-difference equations. With the aid of symbolic computation, we choose the Toda lattice to illustrate the validity and advantages of the generalized work. As a result, 1-soliton, 2-soliton, and 3-soliton solutions are obtained, from which the uniform formula of *N*-soliton solutions is derived. It is shown that the exp-function method may provide us with a straightforward and effective mathematical tool for generating *N*-soliton solutions of nonlinear differential-difference equations in mathematical physics.

Key words: Nonlinear Differential-Difference Equations; Exp-Function Method; *N*-Soliton Solutions.

1. Introduction

Since the work of Fermi et al. in the 1950s [1], the investigation of exact solutions of nonlinear differential-difference equations (NLDDEs) has played a crucial role in the modelling of many phenomena in different fields which include condensed matter physics, biophysics, and mechanical engineering. We can also encounter such systems in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models. Unlike difference equations which are fully discretized, differential-difference equations (DDEs) are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous. The Toda lattice is a simple model for a nonlinear one-dimensional crystal. It describes the motion of a chain of particles with nearest-neighbour interaction [2]. The important property of the Toda equation is the existence of so-called soliton solutions, that is, puls-like waves which spread in time without changing their size or shape and interact with each other in a particle-like way [3]. This is a surprising phenomenon, since for a generic linear equation one would expect spreading of waves (dispersion) and for a generic nonlinear force one would expect that solutions only exist for a finite time (breaking of waves). Obviously our particular force is such that both phenomena cancel each other giving rise to a stable wave existing for all time [4]!

Recently, He and Wu [5] proposed a new method called the exp-function method to seek solitary wave solutions, periodic solutions, and compact-like solutions of nonlinear evolution equations (NLEEs). The basic idea of the exp-function method was presented in He's monograph [6]. The method was used by many researchers to study various NLEEs in the straightforward way or in the sub-equation way, such as the double sine-Gordon equation [7], Burgers' equation [8], Maccari's system [9], the Klein-Gordon equation [10], the combined KdV-mKdV equation [11], variant Boussinesq equations [12], Broer-Kaup-Kupershmidt equations [13], variable-coefficient equations [14 – 16], and high-dimensional equations [17 – 19]. It is Zhu [20 – 22] who first, and the researchers [23 – 25] later, applied the exp-function method to NLDDEs. The studies show that the exp-function method is straightforward, concise, and its applications are promising. More recently, Marinakis [26] did very interesting work to generalize the exp-function method for constructing *N*-soliton solutions of NLEEs. With the generalized work, Marinakis successfully obtained the known 2-soliton and 3-soliton solutions of the famous Korteweg-de Vries (KdV) equation in a simple and straightforward way. Zhang and Zhang [27] improved Marinakis work to obtain 1-soliton, 2-soliton, 3-soliton solutions, and more importantly, the uniform formula of N-soliton solutions of a KdV equations with variable coefficients. In another very interesting work, Ebaid [28] generalized the exp-function method by using a general function, which satisfies the Riccati equation.

In the present paper, we would like to further generalize the exp-function method for constructing N-soliton solutions of NLDDEs. In order to illustrate the effectiveness and advantage of our work, we will consider the Toda lattice in the form

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \mathrm{e}^{x_{n-1}-x_n} - \mathrm{e}^{x_n - x_{n+1}}, \tag{1}$$

where $x_n = x_n(t)$ is the displacement from equilibrium of the nth unit mass under an exponential decaying interaction force between nearest neighbours. If set $y_n = x_n - x_{n+1}$, and hence $y_{n-1} = x_{n-1} - x_n$, from (1) we have

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \mathrm{e}^{y_{n-1}} - \mathrm{e}^{y_n}, \tag{2}$$

$$\frac{\mathrm{d}^2 x_{n+1}}{\mathrm{d}t^2} = \mathrm{e}^{y_n} - \mathrm{e}^{y_{n+1}}, \tag{3}$$

then (1) is reduced to

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = \mathrm{e}^{y_{n-1}} - 2\mathrm{e}^{y_n} + \mathrm{e}^{y_{n+1}}. \tag{4}$$

The rest of this paper is organized as follows. In Section 2, we give the description of the exp-function method for constructing N-soliton solutions of NLDDEs. In Section 3, we apply this method to (4). In Section 4, some conclusions and discussions are given.

2. Basic Idea of the Exp-Function Method for N-Soliton Solutions of NLDDEs

For a given NLDDE, for example

$$\triangle(u_{nt}, u_{nx}, u_{ntt}, u_{nxt}, \ldots u_{n-1}, u_n, u_{n+1}, \ldots) = 0, \tag{5}$$

the exp-function method [5] generalized in this paper to construct 1-soliton solution is based on the assumption that its solutions can be expressed as

$$u_n(x,t) = \frac{\sum_{i_1=0}^{p_1} a_{i_1} \mathrm{e}^{i_1 \xi_1}}{\sum_{j_1=0}^{q_1} b_{j_1} \mathrm{e}^{j_1 \xi_1}}, \tag{6}$$

$$u_{n-s}(x,t) = \frac{\sum_{i_1=0}^{p_1} a_{i_1} \mathrm{e}^{i_1(\xi_1 - sk_1)}}{\sum_{j_1=0}^{q_1} b_{j_1} \mathrm{e}^{j_1(\xi_1 - sk_1)}}, \tag{7}$$

$$u_{n+s}(x,t) = \frac{\sum_{i_1=0}^{p_1} a_{i_1} \mathrm{e}^{i_1(\xi_1 + sk_1)}}{\sum_{j_1=0}^{q_1} b_{j_1} \mathrm{e}^{j_1(\xi_1 + sk_1)}}, \tag{8}$$

where $\xi_1 = k_1 n + l_1 x + c_1 t + w_1$, s is an integral number, a_{i_1}, b_{j_1}, k_1, l_1, c_1, and w_1 are unknown constants, the values of p_1 and q_1 can be determined by balancing the linear term of highest order in (5) with the highest-order nonlinear term.

In order to seek N-soliton solutions for any integer $N > 1$, we generalize (6) – (8) as follows:

$$u_n(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} \mathrm{e}^{\sum_{g=1}^{N} i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N} \mathrm{e}^{\sum_{g=1}^{N} j_g \xi_g}}, \tag{9}$$

$$u_{n-s}(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} \mathrm{e}^{\sum_{g=1}^{N} i_g(\xi_g - sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N} \mathrm{e}^{\sum_{g=1}^{N} j_g(\xi_g - sk_g)}}, \tag{10}$$

$$u_{n+s}(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} \mathrm{e}^{\sum_{g=1}^{N} i_g(\xi_g + sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N} \mathrm{e}^{\sum_{g=1}^{N} j_g(\xi_g + sk_g)}}, \tag{11}$$

where $\xi_g = k_g n + l_g x + c_g t + w_g$. When $N = 2$, (9) – (11) become

$$u_n(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} \mathrm{e}^{\sum_{g=1}^{2} i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} b_{j_1 j_2} \mathrm{e}^{\sum_{g=1}^{2} j_g \xi_g}}, \tag{12}$$

$$u_{n-s}(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} \mathrm{e}^{\sum_{g=1}^{2} i_g(\xi_g - sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} b_{j_1 j_2} \mathrm{e}^{\sum_{g=1}^{2} j_g(\xi_g - sk_g)}}, \tag{13}$$

$$u_{n+s}(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} \mathrm{e}^{\sum_{g=1}^{2} i_g(\xi_g + sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} b_{j_1 j_2} \mathrm{e}^{\sum_{g=1}^{2} j_g(\xi_g + sk_g)}}, \tag{14}$$

which can be used to construct the 2-soliton solution.

When $N = 3$, (9) – (11) become:

$$u_n(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \sum_{i_3=0}^{p_3} a_{i_1 i_2 i_3} \mathrm{e}^{\sum_{g=1}^{3} i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{q_3} b_{j_1 j_2 j_3} \mathrm{e}^{\sum_{g=1}^{3} j_g \xi_g}}, \tag{15}$$

$$u_{n-s}(x,t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \sum_{i_3=0}^{p_3} a_{i_1 i_2 i_3} \mathrm{e}^{\sum_{g=1}^{3} i_g(\xi_g - sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{q_3} b_{j_1 j_2 j_3} \mathrm{e}^{\sum_{g=1}^{3} j_g(\xi_g - sk_g)}}, \tag{16}$$

$$u_{n+s}(x,t)=\frac{\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\sum_{i_3=0}^{p_3}a_{i_1i_2i_3}\mathrm{e}^{\sum_{g=1}^{3}i_g(\xi_g+sk_g)}}{\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\sum_{j_3=0}^{q_3}b_{j_1j_2j_3}\mathrm{e}^{\sum_{g=1}^{3}j_g(\xi_g+sk_g)}}, \tag{17}$$

which can be used to obtain the 3-soliton solution.

Substituting (12) – (14) into (5), and using the Mathematica software, then equating to zero each coefficient of the same order power of the exponential functions yields a set of equations. Solving this set of equations, we can determine the 2-soliton solution and the following 3-soliton solution by means of (15) – (17), provide they exist. If possible, we may conclude with the uniform formula of N-soliton solutions for any $N \geq 1$.

3. Application to the Toda Lattice Equation

Using the transformation

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}=\mathrm{e}^{y_n}-1, \tag{18}$$

we have

$$\mathrm{e}^{y_n}=\frac{\mathrm{d}u_n}{\mathrm{d}t}+1, \tag{19}$$

$$\mathrm{e}^{y_{n-1}}=\frac{\mathrm{d}u_{n-1}}{\mathrm{d}t}+1, \tag{20}$$

$$\mathrm{e}^{y_{n+1}}=\frac{\mathrm{d}u_{n+1}}{\mathrm{d}t}+1. \tag{21}$$

Substituting (19) – (21) into (4) yields

$$\frac{\mathrm{d}^2y_n}{\mathrm{d}t^2}=\frac{\mathrm{d}u_{n-1}}{\mathrm{d}t}-2\frac{\mathrm{d}u_n}{\mathrm{d}t}+\frac{\mathrm{d}u_{n+1}}{\mathrm{d}t}, \tag{22}$$

then integrating (22) with respect to t once and setting the integration constant to zero, we obtain

$$\frac{\mathrm{d}y_n}{\mathrm{d}t}=u_{n-1}-2u_n+u_{n+1}. \tag{23}$$

Differentiating (18) with respect to t and using (19) and (23), we get

$$\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2}=\left(\frac{\mathrm{d}u_n}{\mathrm{d}t}+1\right)(u_{n-1}-2u_n+u_{n+1}). \tag{24}$$

To seek the 1-soliton solution, we suppose:

$$u_n(t)=\frac{a_1\mathrm{e}^{\xi_1}}{1+b_1\mathrm{e}^{\xi_1}}, \tag{25}$$

$$u_{n-1}(t)=\frac{a_1\mathrm{e}^{\xi_1-k_1}}{1+b_1\mathrm{e}^{\xi_1-k_1}}, \tag{26}$$

$$u_{n+1}(t)=\frac{a_1\mathrm{e}^{\xi_1+k_1}}{1+b_1\mathrm{e}^{\xi_1+k_1}}, \tag{27}$$

where $\xi_1=k_1n+c_1t+w_1$. Obviously, (25) – (27) are embedded in the same form as (6) – (8). Substituting (25) – (27) into (24), and using Mathematica, then equating to zero each coefficient of the same order power of the exponential functions yields a set of equations as follows:

$$\begin{aligned}
&-a_1+2a_1\mathrm{e}^{k_1}+a_1c_1^2\mathrm{e}^{k_1}-a_1\mathrm{e}^{2k_1}=0,\\
&-a_1b_1-a_1^2c_1+a_1b_1c_1^2+2a_1b_1\mathrm{e}^{k_1}+2a_1^2c_1\mathrm{e}^{k_1}\\
&-a_1b_1c_1^2\mathrm{e}^{k_1}-a_1b_1\mathrm{e}^{2k_1}-a_1^2c_1\mathrm{e}^{2k_1}+a_1b_1c_1^2\mathrm{e}^{2k_1}=0,\\
&a_1b_1^2+a_1^2b_1c_1-a_1b_1^2c_1^2-2a_1b_1^2\mathrm{e}^{k_1}-2a_1^2b_1c_1\mathrm{e}^{k_1}\\
&+a_1b_1^2c_1^2\mathrm{e}^{k_1}+a_1b_1^2\mathrm{e}^{2k_1}+a_1^2b_1c_1\mathrm{e}^{2k_1}-a_1b_1^2c_1^2\mathrm{e}^{2k_1}=0,\\
&a_1b_1^3-2a_1b_1^3\mathrm{e}^{k_1}-a_1b_1^3c_1^2\mathrm{e}^{k_1}+a_1b_1^3\mathrm{e}^{2k_1}=0.
\end{aligned}$$

Solving this set of equations, we have

$$a_1=2b_1\sinh\frac{k_1}{2},\quad c_1=2\sinh\frac{k_1}{2}. \tag{28}$$

We, therefore, obtain the 1-soliton solution of (4):

$$\begin{aligned}
y_n&=\ln\left[2b_1\sinh\frac{k_1}{2}\left(\frac{\mathrm{e}^{\xi_1}}{1+b_1\mathrm{e}^{\xi_1}}\right)_t+1\right]\\
&=\ln\left\{\left[\ln(1+b_1\mathrm{e}^{\xi_1})\right]_{tt}+1\right\},
\end{aligned} \tag{29}$$

where $\xi_1=k_1n+2\sinh\frac{k_1}{2}t+w_1$, b_1, k_1, and w_1 are arbitrary constants. We note that if setting $b_1=1$, (29) becomes the solution (2.4.75) obtained in [29].

To construct the 2-soliton solution, we suppose

$$u_n(t)=\frac{a_{10}\mathrm{e}^{\xi_1}+a_{01}\mathrm{e}^{\xi_2}+a_{11}\mathrm{e}^{\xi_1+\xi_2}}{1+b_1\mathrm{e}^{\xi_1}+b_2\mathrm{e}^{\xi_2}+b_3\mathrm{e}^{\xi_1+\xi_2}}, \tag{30}$$

$$u_{n-1}(t)=\frac{a_{10}\mathrm{e}^{\xi_1-k_1}+a_{01}\mathrm{e}^{\xi_2-k_2}+a_{11}\mathrm{e}^{\xi_1+\xi_2-k_1-k_2}}{1+b_1\mathrm{e}^{\xi_1-k_1}+b_2\mathrm{e}^{\xi_2-k_2}+b_3\mathrm{e}^{\xi_1+\xi_2-k_1-k_2}}, \tag{31}$$

$$u_{n+1}(t)=\frac{a_{10}\mathrm{e}^{\xi_1+k_1}+a_{01}\mathrm{e}^{\xi_2+k_2}+a_{11}\mathrm{e}^{\xi_1+\xi_2+k_1+k_2}}{1+b_1\mathrm{e}^{\xi_1+k_1}+b_2\mathrm{e}^{\xi_2+k_2}+b_3\mathrm{e}^{\xi_1+\xi_2+k_1+k_2}}, \tag{32}$$

where $\xi_i=k_in+c_it+w_i$ $(i=1,2)$. Clearly, (30) – (32) possess the same form as (12) – (14). Substituting

(30) – (32) into (24), and using the similar manipulations as illustrated above, we get a set of equations. Solving the set of equations (see Appendix 1 for more detailed process), we have

$$a_{10} = 2b_1 \sinh\frac{k_1}{2}, \quad a_{01} = 2b_2 \sinh\frac{k_2}{2},$$
$$a_{11} = 2b_1b_2 e^{B_{12}}\left(\sinh\frac{k_1}{2} + \sinh\frac{k_2}{2}\right), \tag{33}$$

$$b_3 = b_1b_2 e^{B_{12}}, \ c_i = 2\sinh\frac{k_i}{2} \quad (i = 1,2),$$
$$e^{B_{12}} = \frac{\sinh^2\frac{k_1-k_2}{4}}{\sinh^2\frac{k_1+k_2}{4}}, \tag{34}$$

Thus, we obtain the 2-soliton solution of (4):

$$\begin{aligned} y_n &= \ln\left\{2\left[\frac{b_1\sinh\frac{k_1}{2}e^{\xi_1} + b_2\sinh\frac{k_2}{2}e^{\xi_2} + b_1b_2\left(\sinh\frac{k_1}{2}+\sinh\frac{k_2}{2}\right)e^{\xi_1+\xi_2+B_{12}}}{1+b_1e^{\xi_1}+b_2e^{\xi_2}+b_1b_2e^{\xi_1+\xi_2+B_{12}}}\right]_t + 1\right\} \\ &= \ln\left\{\left[\ln(1+b_1e^{\xi_1}+b_2e^{\xi_2}+b_1b_2e^{\xi_1+\xi_2+B_{12}})\right]_{tt} + 1\right\}, \end{aligned} \tag{35}$$

where $\xi_i = k_i n + 2\sinh\frac{k_i}{2}t + w_i$ $(i = 1,2)$, b_1, b_2, k_1, k_2, w_1, and w_2 are free constants, $e^{B_{12}}$ is defined by (34). If setting $b_1 = b_2 = 1$, (35) changes into the solution (2.4.81) obtained in [29]. We now construct the 3-soliton solution, for this purpose, we suppose

$$u_n(x) = \frac{f_{1,n}(\xi_1,\xi_2,\xi_3)}{f^2_{2,n}(\xi_1,\xi_2,\xi_3)}, \tag{36}$$

$$u_{n-1}(x) = \frac{f_{1,n-1}(\xi_1,\xi_2,\xi_3)}{f^2_{2,n-1}(\xi_1,\xi_2,\xi_3)}, \tag{37}$$

$$u_{n+1}(x) = \frac{f_{1,n+1}(\xi_1,\xi_2,\xi_3)}{f^2_{2,n+1}(\xi_1,\xi_2,\xi_3)}, \tag{38}$$

where $\xi_i = k_i n + c_i t + w_i$ $(i = 1,2,3)$, and

$$f_{1,n}(\xi_1,\xi_2,\xi_3) = a_{100}e^{\xi_1} + a_{010}e^{\xi_2} + a_{001}e^{\xi_3} + a_{110}e^{\xi_1+\xi_2} + a_{101}e^{\xi_1+\xi_3} + a_{011}e^{\xi_2+\xi_3} + a_{111}e^{\xi_1+\xi_2+\xi_3},$$

$$f_{2,n}(\xi_1,\xi_2,\xi_3) = 1 + b_1e^{\xi_1} + b_2e^{\xi_2} + b_3e^{\xi_3} + b_4e^{\xi_1+\xi_2} + b_5e^{\xi_1+\xi_3} + b_6e^{\xi_2+\xi_3} + b_7e^{\xi_1+\xi_2+\xi_3},$$

$$\begin{aligned} f_{1,n-1}(\xi_1,\xi_2,\xi_3) = {} & a_{100}e^{\xi_1-k_1} + a_{010}e^{\xi_2-k_2} + a_{001}e^{\xi_3-k_3} + a_{110}e^{\xi_1+\xi_2-k_1-k_2} \\ & + a_{101}e^{\xi_1+\xi_3-k_1-k_3} + a_{011}e^{\xi_2+\xi_3-k_2-k_3} + a_{111}e^{\xi_1+\xi_2+\xi_3-k_1-k_2-k_3}, \end{aligned}$$

$$\begin{aligned} f_{2,n-1}(\xi_1,\xi_2,\xi_3) = {} & 1 + b_1e^{\xi_1-k_1} + b_2e^{\xi_2-k_2} + b_3e^{\xi_3-k_3} + b_4e^{\xi_1+\xi_2-k_1-k_2} \\ & + b_5e^{\xi_1+\xi_3-k_1-k_3} + b_6e^{\xi_2+\xi_3-k_2-k_3} + b_7e^{\xi_1+\xi_2+\xi_3-k_1-k_2-k_3}, \end{aligned}$$

$$\begin{aligned} f_{1,n+1}(\xi_1,\xi_2,\xi_3) = {} & a_{100}e^{\xi_1+k_1} + a_{010}e^{\xi_2+k_2} + a_{001}e^{\xi_3+k_3} + a_{110}e^{\xi_1+\xi_2+k_1+k_2} \\ & + a_{101}e^{\xi_1+\xi_3+k_1+k_3} + a_{011}e^{\xi_2+\xi_3+k_2+k_3} + a_{111}e^{\xi_1+\xi_2+\xi_3+k_1+k_2+k_3}, \end{aligned}$$

$$\begin{aligned} f_{2,n+1}(\xi_1,\xi_2,\xi_3) = {} & 1 + b_1e^{\xi_1+k_1} + b_2e^{\xi_2+k_2} + b_3e^{\xi_3+k_3} + b_4e^{\xi_1+\xi_2+k_1+k_2} \\ & + b_5e^{\xi_1+\xi_3+k_1+k_3} + b_6e^{\xi_2+\xi_3+k_2+k_3} + b_7e^{\xi_1+\xi_2+\xi_3+k_1+k_2+k_3}. \end{aligned}$$

It is easy to see that (36) – (38) have the same form as (15) – (17). By the similar manipulations mentioned above, we have

$$a_{100} = 2b_1\sinh\frac{k_1}{2}, \quad a_{010} = 2b_2\sinh\frac{k_2}{2},$$
$$a_{001} = 2b_3\sinh\frac{k_3}{2}, \tag{39}$$

$$a_{110} = 2b_1b_2e^{B_{12}}\left(\sinh\frac{k_1}{2} + \sinh\frac{k_2}{2}\right),$$
$$a_{101} = 2b_1b_3e^{B_{13}}\left(\sinh\frac{k_1}{2} + \sinh\frac{k_3}{2}\right), \tag{40}$$
$$a_{011} = 2b_2b_3e^{B_{23}}\left(\sinh\frac{k_2}{2} + \sinh\frac{k_3}{2}\right),$$

$$a_{111} = 2b_1b_2b_3 e^{B_{12}+B_{13}+B_{23}} \cdot (\sinh\frac{k_1}{2} + \sinh\frac{k_2}{2} + \sinh\frac{k_3}{2}), \tag{41}$$

$$b_4 = b_1b_2 e^{B_{12}}, \quad b_5 = b_1b_3 e^{B_{13}}, \quad b_6 = b_2b_3 e^{B_{23}}, \quad b_7 = b_1b_2b_3 e^{B_{12}+B_{13}+B_{23}}, \tag{42}$$

$$c_i = 2\sinh\frac{k_i}{2} \quad (i = 1,2,3), \quad e^{B_{ij}} = \frac{\sinh^2\frac{k_i-k_j}{4}}{\sinh^2\frac{k_i+k_j}{4}} \quad (1 \le i < j \le 3). \tag{43}$$

Employing (39) – (43), we obtain the 3-soliton solution of (4):

$$y_n = \ln\Big\{\Big[\ln(1 + b_1e^{\xi_1} + b_2e^{\xi_2} + b_3e^{\xi_3} + b_1b_2e^{\xi_1+\xi_2+B_{12}} + b_1b_3e^{\xi_1+\xi_3+B_{13}} + b_2b_3e^{\xi_2+\xi_3+B_{23}} + b_1b_2b_3e^{\xi_1+\xi_2+\xi_3+B_{12}+B_{13}+B_{23}})\Big]_{tt} + 1\Big\}, \tag{44}$$

where $\xi_i = k_in + 2\sinh\frac{k_i}{2}t + w_i$ $(i = 1,2,3)$, b_1, b_2, b_3, k_1, k_2, k_3, w_1, w_2, and w_3 are arbitrary constants, B_{ij} $(1 \le i < j \le 3)$ are determined by (43).

If we continue to construct the N-soliton solutions for any $N \ge 4$, the following similar manipulations becomes rather complicated since equating to zero the coefficients of the exponential functions implies a highly nonlinear system as pointed out in [26]. Fortunately, by analyzing the obtained solutions (29), (35), and (44) we can obtain the uniform formula of N-soliton solutions as follows:

$$y_n = \ln\Big\{\Big[\ln\Big(\sum_{\mu=0,1}\prod_{i=1}^{N} b_i^{\mu_i} e^{\sum_{i=1}^{N}\mu_i\xi_i + \sum_{1\le i<j\le N}\mu_i\mu_jB_{ij}}\Big)\Big]_{tt} + 1\Big\}, \tag{45}$$

where the summation $\sum_{\mu=0,1}$ refers to all combinations of each $\mu_i = 0,1$ for $i = 1,2,\cdots,N$, $\xi_i = k_in + 2\sinh\frac{k_i}{2}t + w_i$, $e^{B_{ij}} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}$ $(i < j; i,j = 1,2,\cdots,N)$. If set $b_1 = b_2 = b_3 = 1$, solution (45) becomes the solution (2.4.84) obtained in [29].

The asymptotic properties of a single soliton going to the left, two overtaking solitons with the higher soliton catching up the lower one and the collisions among three overtaking solitons are shown in Figures 1 – 3, respectively.

Different from the continuous case, each exotic wave in Figures 4 – 6 does not exhibit a singular property, although solutions (29), (33), and (45) possess singular points when $b_1 = -1$, $b_2 = -1$. Such exotic waves with a similar property were found in [30].

Remark 1. Solutions (29), (35), and (44) obtained above have been checked with Mathematica by putting them back into the original equation (4). According to the Hirota bilinear method [29], solution (45) can also be proved. To the best of our knowledge, solutions (29), (35), (44), and (45) with arbitrary constants have not been reported in literatures.

4. Conclusions and Discussions

In summary, 1-soliton, 2-soliton, and 3-soliton solutions of the Toda lattice equation have been obtained, from which the uniform formula of N-soliton solutions is derived. This is due to the generalization of the exp-function method presented in this paper. Though the N-soliton solutions (45) can be constructed by the Hirota bilinear method [29], the proposed method with the help of Mathematica for generating 1-soliton, 2-soliton, and 3-soliton solutions (29), (35), and (44) is more simple and straightforward. In a general way, when we use the Hirota bilinear method, the following steps will be taken. Firstly, the considered equation must be reduced to the so-called Hirota bilinear form of one or more new dependent variables by means of a suitable transformation and the defined bilinear operator. Secondly, expanding each of the new dependent variables in infinite series of a formal expansion parameter, we split the Hirota bilinear form into a system of linear differential equations, from which we truncate the infinite series by selecting some appropriate exponential function solutions of the obtained differential equations. Finally, we use the selected exponential function solutions to determine the new variables and, hence, the multi-soliton solutions of the given equation. Compared with the Hirota bilinear method, the exp-function method does not take above steps in constructing N-solition solutions. In this sense, we may conclude that the exp-function method has the advantage of simplicity and effectiveness and may provide us with a straightforward and applicable mathematical tool for generating N-soliton solutions or testing their existence, and it can be extended to other NLDDEs in mathematical physics.

Recently, Kudryashov [31, 32] and Kudryashov and Loguinova [33] presented some suggestions to the exp-function method. In our opinions, we have our experience and sympathy to this method. As a kind of direct

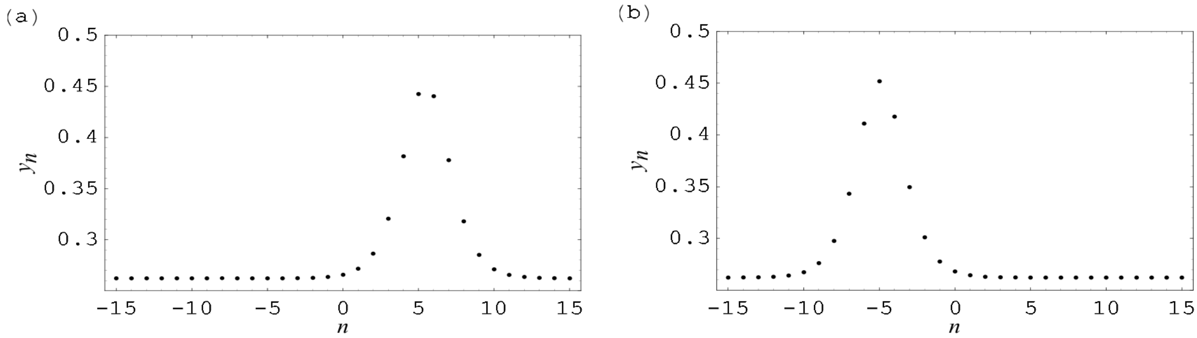

Fig. 1. Evolution plots of the 1-soliton solution (29) with parameters $b_1 = 1$, $k_1 = 1.3$, $w_1 = 0$ at times: (a) $t = -5$; (b) $t = 5$.

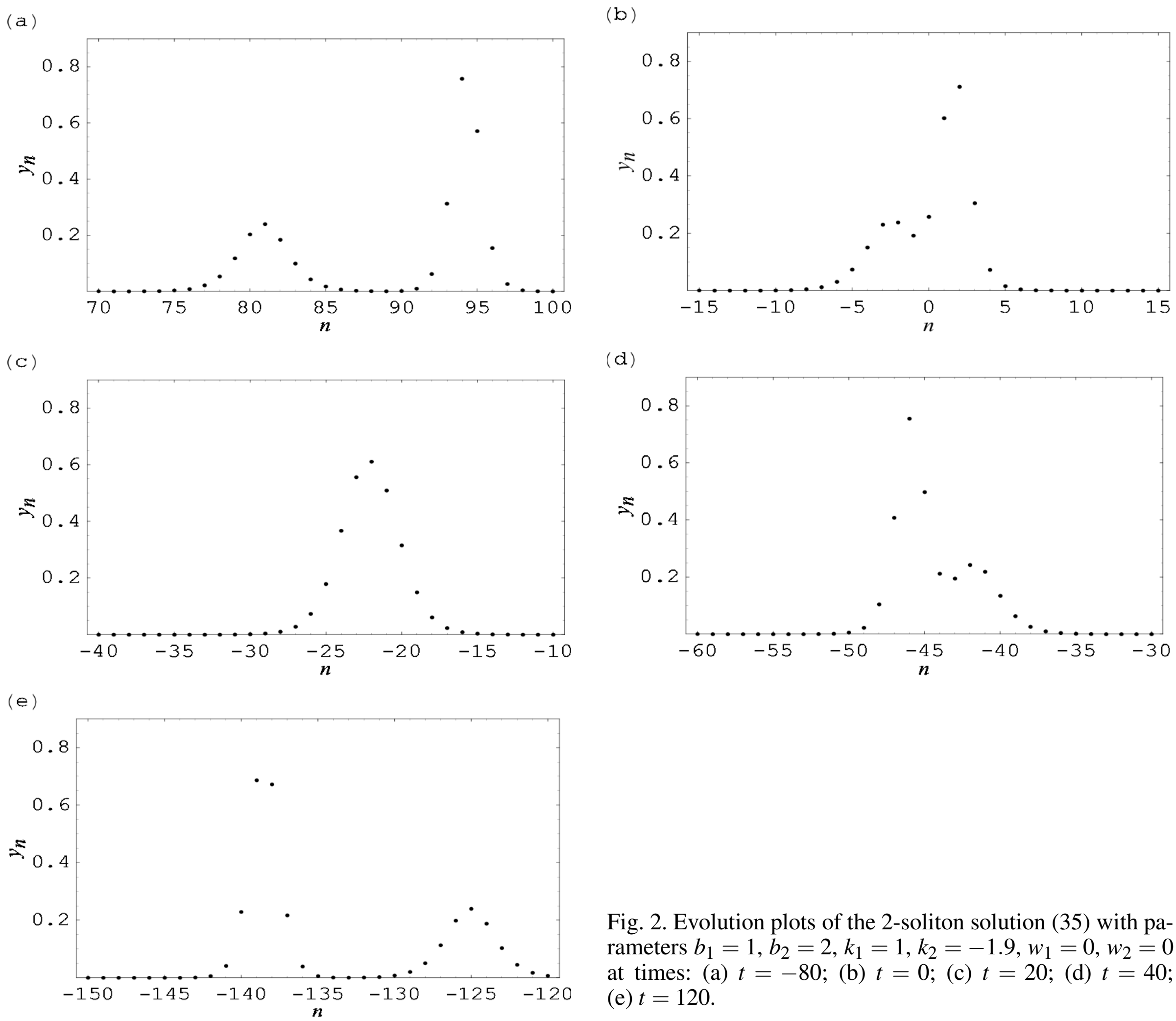

Fig. 2. Evolution plots of the 2-soliton solution (35) with parameters $b_1 = 1$, $b_2 = 2$, $k_1 = 1$, $k_2 = -1.9$, $w_1 = 0$, $w_2 = 0$ at times: (a) $t = -80$; (b) $t = 0$; (c) $t = 20$; (d) $t = 40$; (e) $t = 120$.

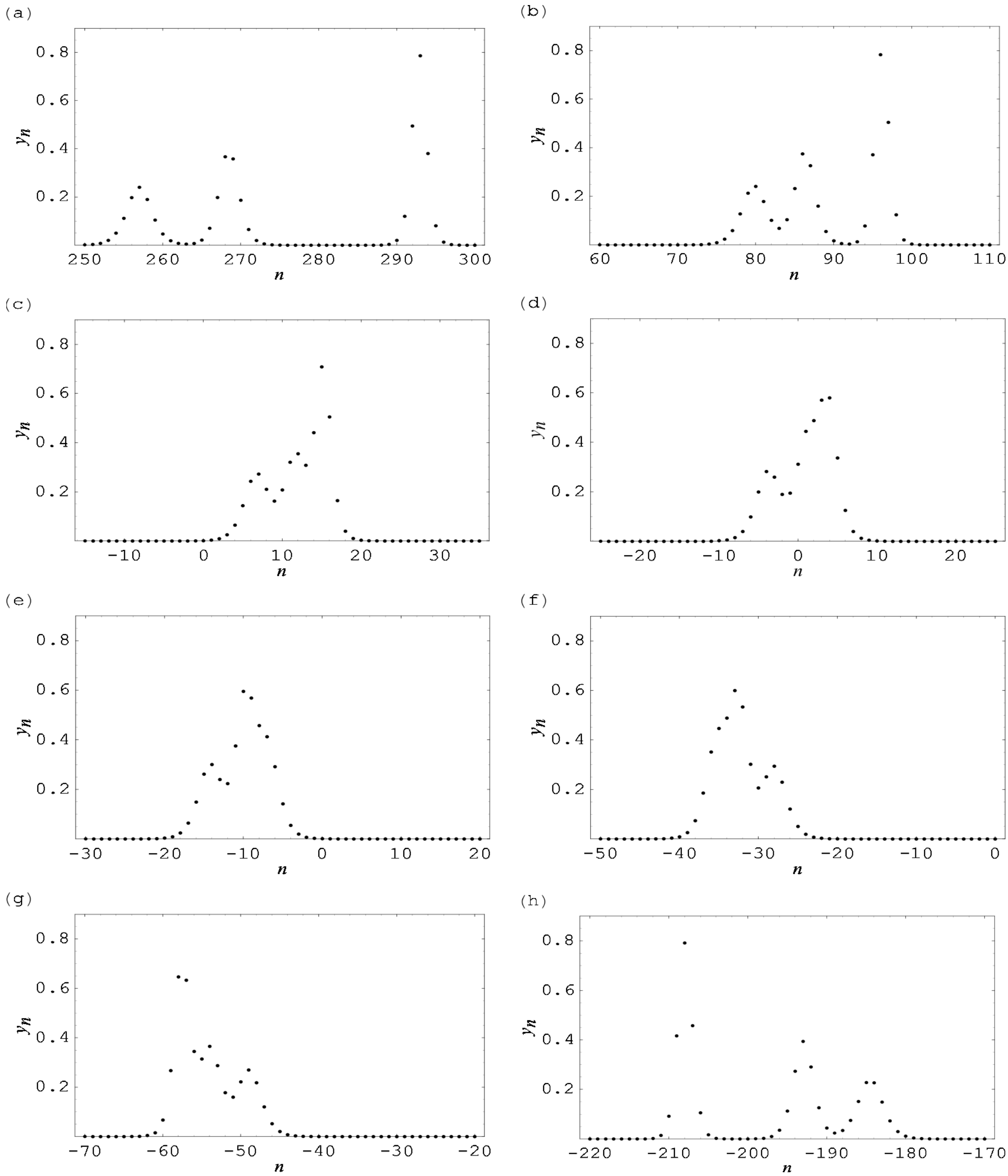

Fig. 3. Evolution plots of the 3-soliton solution (44) with parameters $b_1 = 1$, $b_2 = 2$, $b_3 = 2$, $k_1 = 1.3$, $k_2 = -1.9$, $k_3 = 1$, $w_1 = 0$, $w_2 = 0$, $w_3 = 0$ at times: (a) $t = -250$; (b) $t = -80$; (c) $t = -10$; (d) $t = 0$; (e) $t = 10$; (f) $t = 30$; (g) $t = 50$; (h) $t = 80$.

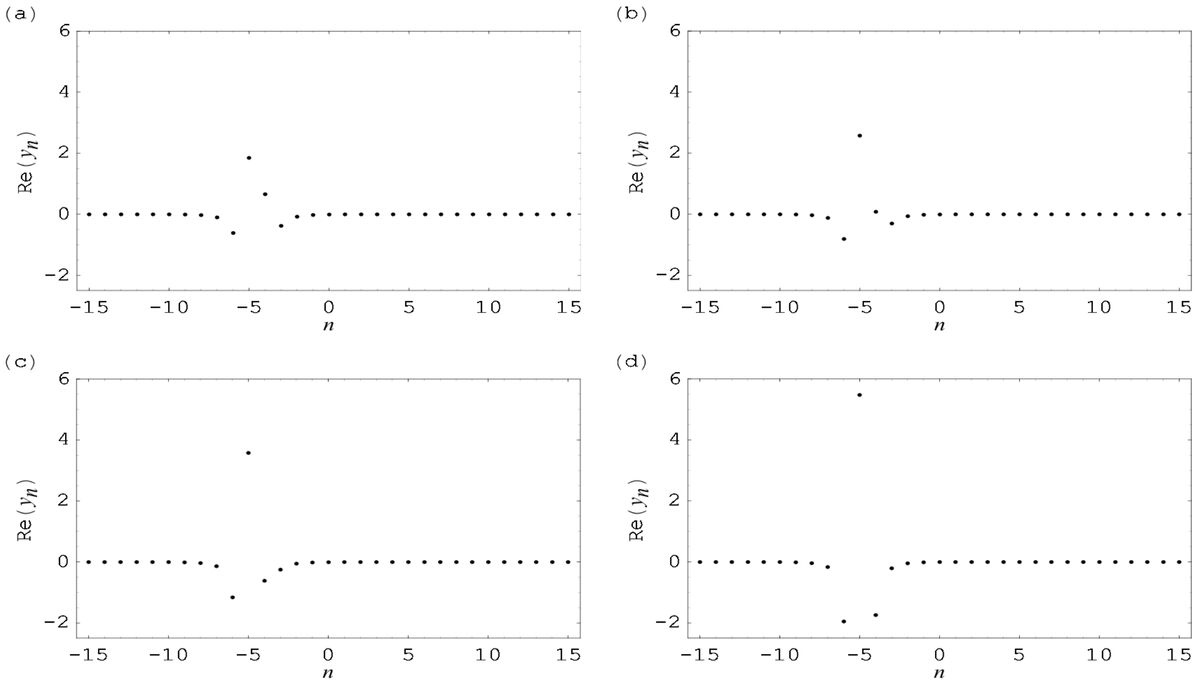

Fig. 4. Evolution plots of the real part of the 1-soliton solution (29) with parameters $b_1 = -1$, $k_1 = 1.3$, $w_1 = 0$ at times: (a) $t = 4.3$; (b) $t = 4.4$; (c) $t = 4.5$; (d) $t = 4.6$.

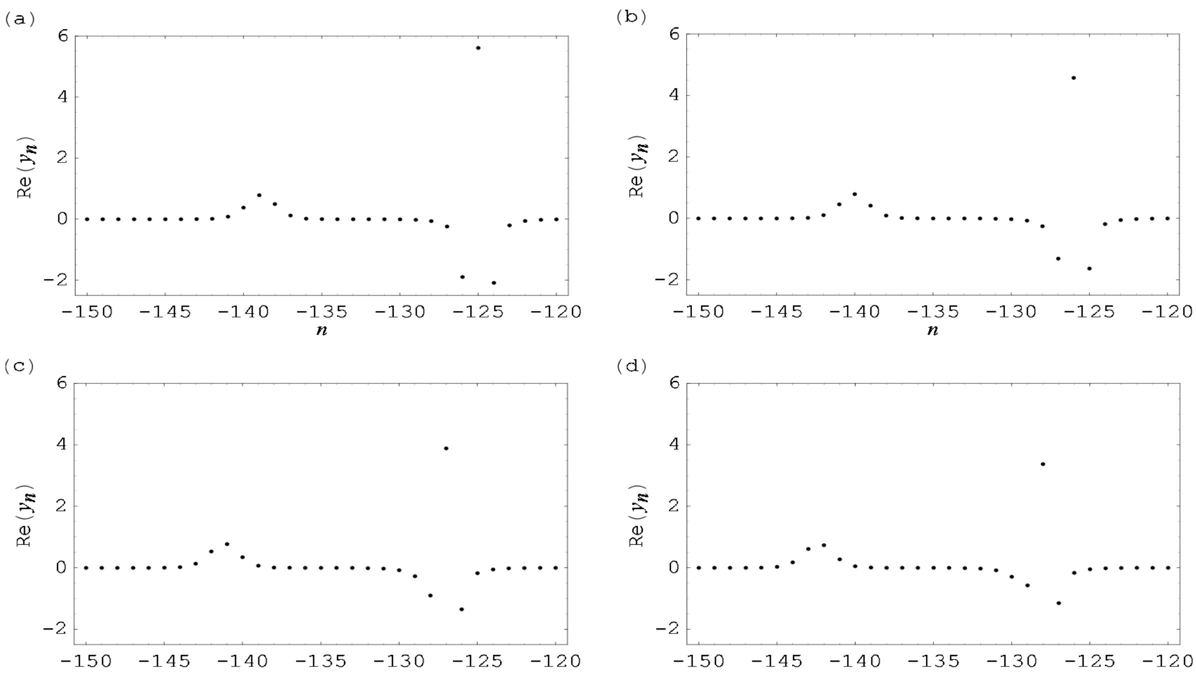

Fig. 5. Evolution plots of the real part of the 2-soliton solution (35) with parameters $b_1 = -1$, $b_2 = 1$, $k_1 = 1$, $k_2 = -1.9$, $w_1 = 0$, $w_2 = 0$ at times: (a) $t = 120$; (b) $t = 121$; (c) $t = 122$; (d) $t = 123$.

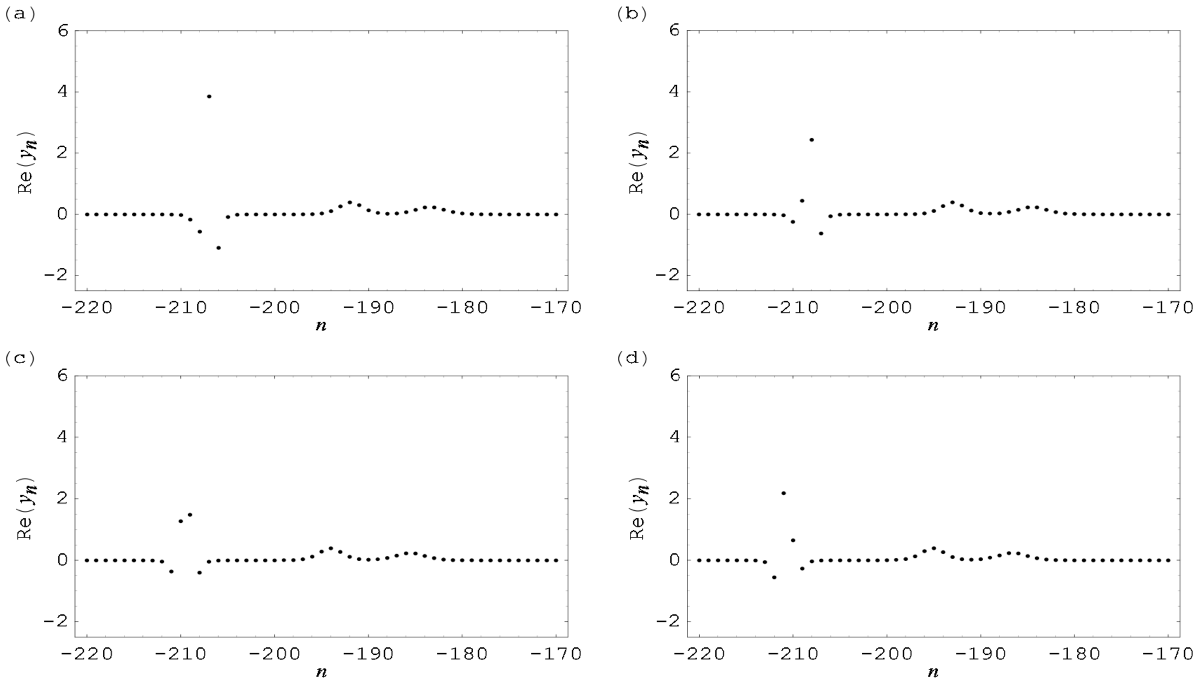

Fig. 6. Evolution plots of the real part of the 3-soliton solution (44) with parameters $b_1 = 1$, $b_2 = -1$, $b_3 = 2$, $k_1 = 1.3$, $k_2 = -1.9$, $k_3 = 1$, $w_1 = 0$, $w_2 = 0$, $w_3 = 0$ at times: (a) $t = 179$; (b) $t = 180$; (c) $t = 181$; (d) $t = 182$.

method, the exp-function method can be used to find solitary wave solutions, periodic solutions, and compact-like solutions of NLEEs by a unified ansatz, i. e. Equation (4) in [5]. Moreover, the exp-function method has been generalized to construct *N*-solition solutions of NLEEs [26] and NLDDEs as did in this paper. Generally speaking, it is hard even impossible to generalize one method for NLEEs to solve NLDDEs because of the difficulty in finding iterative relations from indices n to $n \pm 1$, $n \pm 2$, etc. Fortunately, the exp-function method, seen from the work [22], overcomes such difficulty and can be generalized to solve NLDDEs owing to the properties of the exponential function. The tanh-function method cannot obtain *N*-solition solutions for either NLEEs or NLDDEs. In this meaning, we think that the exp-function method is more general than the tanh-function method. Similar to other direct methods including the tanh-function method, the main idea of the exp-function method is to transform the given equation into a set of algebraic equations and solve the undetermined coefficients in the ansatz. It shows that the exp-function method is easier and more concise than the inverse scattering transform and the Hirota bilinear method since they carry out relatively complicated transformations [34]. In [33], the modified simplest equation method was presented and applied to a generalized Korteweg–de Vries equation with source. As a result, some exact solutions were found through solving a third-order linear ordinary differential equation derived from the generalized Korteweg-de Vries equation. The modified simplest equation method is very effective and significant. However, for the physical or engineering researchers, they are better at solving algebraic equation problems than differential equation problems [34]. New special polynomials associated with rational solutions were introduced in [35]. The special polynomials are useful to get rational solutions of some high-order nonlinear differential equations. However, we are usually interested in the solitary solutions and periodic solutions or compact-like solutions. Such kinds of solutions maybe are useful for the investigation of some physical processes in the real world. Certainly, there is no single best method to search exact solutions of NLEEs. As an objective appraisal, the exp-function method has sometimes also its deficiencies, i. e. obtaining relatively complex algebraic equations during the solving procedure [34] and losing some special solutions [33] obtained by the modified simplest equation method. In view of the cumbersome computation in the exp-function method, we can perform it with the help of symbolic computation systems like Mathematica and Maple. To assure the correctness of the obtained solutions, it is necessary to substitute these solutions into the original equations. It is not encouraged to casually assert to have obtained some "new" solutions without checking them carefully [34]. As pointed out in [31], the simple and powerful tool to judge whether the solutions are new is to plot the graphs of the expressions obtained. The expressions that have the same graphs usually are equivalent. Therefore, we think that the exp-function method is an alternative tool for finding exact solutions of NLEEs and NLDDEs in mathematical physics as long as one uses it correctly and carefully.

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Appendix

There are totally 32 equations in the set of equations and some of them are complicated, nonetheless they do not bring difficulty in solving the set of equations. To begin with, if we consider the following simple and central equations:

$$-a_{10}e^{k_1}e^{2k_2}+2a_{10}e^{2k_1}e^{2k_2}+a_{10}c_1^2e^{2k_1}e^{2k_2}$$
$$-a_{10}e^{3k_1}e^{2k_2}=0,$$

$$-a_{10}b_1e^{k_1}e^{2k_2}-a_{10}^2c_1e^{k_1}e^{2k_2}+a_{10}b_1c_1^2e^{k_1}e^{2k_2}$$
$$+2a_{10}b_1e^{2k_1}e^{2k_2}+2a_{10}^2c_1e^{2k_1}e^{2k_2}-a_{10}b_1c_1^2e^{2k_1}e^{2k_2}$$
$$-a_{10}b_1e^{2k_1}e^{2k_2}-a_{10}^2c_1e^{3k_1}e^{2k_2}+a_{10}b_1c_1^2e^{3k_1}e^{2k_2}=0,$$

$$a_{10}b_1^2e^{k_1}e^{2k_2}+a_{10}^2b_1c_1e^{k_1}e^{2k_2}-a_{10}b_1^2c_1^2e^{k_1}e^{2k_2}$$
$$-2a_{10}b_1^2e^{2k_1}e^{2k_2}-2a_{10}^2b_1c_1e^{2k_1}e^{2k_2}+a_{10}b_1^2c_1^2e^{2k_1}e^{2k_2}$$
$$+a_{10}b_1^2e^{3k_1}e^{2k_2}+a_{10}^2b_1c_1e^{3k_1}e^{2k_2}$$
$$-a_{10}b_1^2c_1^2e^{3k_1}e^{2k_2}=0,$$

$$a_{10}b_1^3e^{k_1}e^{2k_2}-2a_{10}b_1^3e^{2k_1}e^{2k_2}-a_{10}b_1^3c_1^2e^{2k_1}e^{2k_2}$$
$$+a_{10}b_1^3e^{3k_1}e^{2k_2}=0,$$

$$-a_{01}e^{2k_1}e^{k_2}+2a_{01}e^{2k_1}e^{2k_2}+a_{01}c_2^2e^{2k_1}e^{2k_2}$$
$$-a_{01}e^{2k_1}e^{3k_2}=0,$$

$$-a_{01}b_2e^{2k_1}e^{k_2}-a_{01}^2c_2e^{2k_1}e^{k_2}+a_{01}b_2c_2^2e^{2k_1}e^{k_2}$$
$$+2a_{01}b_2e^{2k_1}e^{2k_2}+2a_{01}^2c_2e^{2k_1}e^{2k_2}-a_{01}b_2c_2^2e^{2k_1}e^{2k_2}$$
$$-a_{01}b_2e^{2k_1}e^{3k_2}-a_{01}^2c_2e^{2k_1}e^{3k_2}+a_{01}b_2c_2^2e^{2k_1}e^{3k_2}=0,$$

$$a_{01}b_2^2e^{2k_1}e^{k_2}+a_{01}^2b_2c_2e^{2k_1}e^{k_2}-a_{01}b_2^2c_2^2e^{2k_1}e^{k_2}$$
$$-2a_{01}b_2^2e^{2k_1}e^{2k_2}-2a_{01}^2b_2c_2e^{2k_1}e^{2k_2}+a_{01}b_2^2c_2^2e^{2k_1}e^{2k_2}$$
$$+a_{01}b_2^2e^{2k_1}e^{3k_2}+a_{01}^2b_2c_2e^{2k_1}e^{3k_2}$$
$$-a_{01}b_2^2c_2^2e^{2k_1}e^{3k_2}=0,$$

$$a_{01}b_2^3e^{2k_1}e^{k_2}-2a_{01}b_2^3e^{2k_1}e^{2k_2}-a_{01}b_2^3c_2^2e^{2k_1}e^{2k_2}$$
$$+a_{01}b_2^3e^{2k_1}e^{3k_2}=0,$$

$$-a_{11}e^{k_1}e^{k_2}-3a_{01}b_1e^{2k_1}e^{k_2}+2a_{10}b_2e^{2k_1}e^{k_2}$$
$$-a_{01}a_{10}c_1e^{2k_1}e^{k_2}+a_{10}b_2c_1^2e^{2k_1}e^{k_2}-a_{01}b_1e^{3k_1}e^{k_2}$$
$$-a_{10}b_2e^{3k_1}e^{k_2}+2a_{01}b_1e^{k_1}e^{2k_2}-3a_{10}b_2e^{k_1}e^{2k_2}$$
$$-a_{01}a_{10}c_2e^{k_1}e^{2k_2}+a_{01}b_1c_2^2e^{k_1}e^{2k_2}+2a_{11}e^{2k_1}e^{2k_2}$$
$$+4a_{01}b_1e^{2k_1}e^{2k_2}+4a_{10}b_2e^{2k_1}e^{2k_2}+2a_{01}a_{10}c_1e^{2k_1}e^{2k_2}$$
$$+a_{11}c_1^2e^{2k_1}e^{2k_2}-a_{01}b_1c_1^2e^{2k_1}e^{2k_2}+2a_{10}b_2c_1^2e^{2k_1}e^{2k_2}$$
$$+2a_{01}a_{10}c_2e^{2k_1}e^{2k_2}+2a_{11}c_1c_2e^{2k_1}e^{2k_2}$$
$$-2a_{01}b_1c_1c_2e^{2k_1}e^{2k_2}-2a_{10}b_2c_1c_2e^{2k_1}e^{2k_2}$$
$$+a_{11}c_2^2e^{2k_1}e^{2k_2}+2a_{01}b_1c_2^2e^{2k_1}e^{2k_2}-a_{10}b_2c_2^2e^{2k_1}e^{2k_2}$$
$$+2a_{01}b_1e^{3k_1}e^{2k_2}-3a_{10}b_2e^{3k_1}e^{2k_2}-a_{01}a_{10}c_2e^{3k_1}e^{2k_2}$$
$$+a_{01}b_1c_2^2e^{3k_1}e^{2k_2}-a_{01}b_1e^{k_1}e^{3k_2}-a_{10}b_2e^{k_1}e^{3k_2}$$
$$-3a_{01}b_1e^{2k_1}e^{3k_2}+2a_{10}b_2e^{2k_1}e^{3k_2}-a_{01}a_{10}c_1e^{2k_1}e^{3k_2}$$
$$+a_{10}b_2c_1^2e^{2k_1}e^{3k_2}-a_{11}e^{3k_1}e^{3k_2}=0,$$

then the following results are obtained

$$c_1=e^{\frac{1}{2}k_1}-e^{-\frac{1}{2}k_1},\quad c_2=e^{\frac{1}{2}k_2}-e^{-\frac{1}{2}k_2},$$
$$a_{10}=b_1(e^{\frac{1}{2}k_1}-e^{-\frac{1}{2}k_1}),\quad a_{01}=b_2(e^{\frac{1}{2}k_2}-e^{-\frac{1}{2}k_2}),$$
$$a_{11}=\frac{b_1b_2(e^{\frac{1}{2}k_1}-e^{\frac{1}{2}k_2})^2(e^{\frac{1}{2}k_1}+e^{\frac{1}{2}k_2})}{e^{k_1+k_2}-e^{\frac{1}{2}(k_1+k_2)}}.$$

Next, we substitute these obtained results into each of the other equations, one of the complicated equations is simplified as

$$\frac{1}{e^{\frac{1}{2}(k_1+k_2)}-1}b_1^3e^{k_2}(e^{k_1}+1)(e^{k_2}-1)(e^{\frac{1}{2}k_1}-e^{\frac{1}{2}k_2})$$
$$\cdot(e^{k_1}-1)^2[b_1b_2(e^{\frac{1}{2}k_1}-e^{\frac{1}{2}k_2})^2-b_3(e^{\frac{1}{2}(k_1+k_2)}-1)^2]=0,$$

from which we have

$$b_3 = \frac{b_1 b_2 (\mathrm{e}^{\frac{1}{2}k_1} - \mathrm{e}^{\frac{1}{2}k_2})^2}{(\mathrm{e}^{\frac{1}{2}(k_1+k_2)} - 1)^2}.$$

Substituting c_1, c_2, a_{10}, a_{01}, a_{11}, and b_3 obtained above into the other equations tells that they are solutions of the set of equations. We finally rewrite c_1, c_2, a_{10}, a_{01}, a_{11}, and b_3 as the form in (33) and (34) by using the properties of hyperbolic sine function.

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